HOMOMORPHISMS, AMENABILITY AND WEAK AMENABILITY OF BANACH ALGEBRAS

M. ESHAGHI GORDJI

Abstract. In this paper we find some necessary and sufficient conditions for a Banach algebra to be amenable or weakly amenable, by applying the homomorphisms on Banach algebras.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra and let X be a Banach $\mathcal{A}$-bimodule. Then X^* is a Banach $\mathcal{A}$-bimodule if for each $a \in \mathcal{A}$, $x \in X$ and $x^* \in X^*$ we define

$$\langle x, ax^* \rangle = \langle xa, x^* \rangle, \qquad \langle x, x^*a \rangle = \langle ax, x^* \rangle.$$

Let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a Banach algebra homomorphism, then $\mathcal{B}$ is a $\mathcal{A}$-bimodule by the following module actions

$$a.b = \varphi(a)b, \qquad b.a = b\varphi(a) \qquad\qquad (a \in \mathcal{A}, b \in \mathcal{B}).$$

We denote $\mathcal{B}_\varphi$ the above $\mathcal{A}$-bimodule. For a Banach algebra $\mathcal{A}$, $\mathcal{A}^{**}$ with the first Arens product is a Banach algebra. Let X be a Banach $\mathcal{A}-$module, we can extend the actions of $\mathcal{A}$ on X to actions of $\mathcal{A}^{**}$ on X^{**} via

$$a''.x'' = w^*\text{-}\lim_i \lim_j a_i \, x_j$$

and

$$x''.a'' = w^*\text{-}\lim_j \lim_i x_j \, a_i,$$

where $a'' = w^*\text{-}\lim_i a_i$, $x'' = w^*\text{-}\lim_j x_j$.

If X is a Banach $\mathcal{A}$-bimodule then a derivation from $\mathcal{A}$ into X is a continuous linear map D, such that for every $a, b \in \mathcal{A}$, $D(ab) = D(a).b + a.D(b)$. If $x \in X$, and we define $\delta_x : \mathcal{A} \longrightarrow X$ by $\delta_x(a) = a.x - x.a \quad (a \in \mathcal{A})$, then δ_x is a derivation, derivations of this form are called inner derivations. A Banach algebra $\mathcal{A}$ is amenable if every derivation from $\mathcal{A}$ into each dual $\mathcal{A}$-bimodule is inner; i.e. $H^1(\mathcal{A}, X^*) = \{o\}$, this definition was introduced by B. E. Johnson in [7]. $\mathcal{A}$ is weakly amenable if $H^1(\mathcal{A}, \mathcal{A}^*) = \{0\}$, where $H^1(\mathcal{A}, \mathcal{A}^*)$ is the first cohomology group from $\mathcal{A}$ with coefficients in $\mathcal{A}^*$. Bade, Curtis and Dales have introduced the concept of weak amenability for commutative Banach algebras [1]. In this paper we show that for amenability of Banach algebra $\mathcal{A}$, it is enough to show that for every Banach algebra $\mathcal{B}$ and every injective

2000 *Mathematics Subject Classification.* 46HXX.

Key words and phrases. Derivation , Homomorphism.

homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$, $H^1(\mathcal{A}, {\mathcal{B}_\varphi}^*) = \{o\}$. So we introduce two new notations in amenability of Banach algebras and we related this notations to weak amenability.

2. Amenability

Let $\mathcal{A}$ be a Banach algebra and X be a Banach $\mathcal{A}$-bimodule, then $X \oplus_1 \mathcal{A}$ is a Banach space, with the following norm

$$\|(x,a)\| = \|x\| + \|a\| \qquad (a \in \mathcal{A}\ ,\ x \in X).$$

So $X \oplus_1 \mathcal{A}$ is a Banach algebra with the product

$$(x_1, a_1)(x_2, a_2) = (x_1 \cdot a_2 + a_1 \cdot x_2, a_1 a_2)\ .$$

$X \oplus_1 \mathcal{A}$ is called a module extension Banach algebra. It is easy to show that $(X \oplus_1 \mathcal{A})^* = X^* \oplus \mathcal{A}^*$, where this sum is $\mathcal{A}$-bimodule $l_\infty-$sum. In this section we use module extension Banach algebras to finding an easy equivalent condition for amenability of a Banach algebra.

Theorem 2.1. Let $\mathcal{A}$ be a Banach algebra. Then the following assertions are equivalent:
(i) $\mathcal{A}$ is amenable.
(ii) For every Banach algebra $\mathcal{B}$ and every homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$, $H^1(\mathcal{A}, {\mathcal{B}_\varphi}^*) = \{o\}$.
(iii) For every Banach algebra $\mathcal{B}$ and every injective homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$, $H^1(\mathcal{A}, {\mathcal{B}_\varphi}^*) = \{o\}$.
(iv) For every Banach algebra $\mathcal{B}$ and every injective homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$, if $d_\varphi : \mathcal{A} \longrightarrow {\mathcal{B}_\varphi}^*$ is a (bounded) derivation satisfies

$$\langle d_\varphi(a), \varphi(b)\rangle + \langle d_\varphi(b), \varphi(a)\rangle = 0 \qquad (a, b \in \mathcal{A}),$$

then d_φ is inner derivation.
(v) For every Banach algebra $\mathcal{B}$ and every injective homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$, $H^1(\mathcal{A}, {\mathcal{B}_\varphi}^{**}) = \{o\}$.

Proof. The proofs of $(i) \Longrightarrow (ii)$, $(i) \Longrightarrow (v)$, $(ii) \Longrightarrow (iii)$ and $(iii) \Longrightarrow (iv)$ are immediate. We show that $(iv) \Longrightarrow (i)$ and $(v) \Longrightarrow (i)$ hold. Suppose that (iv) holds and let X be a Banach $\mathcal{A}$-bimodule and $D : \mathcal{A} \longrightarrow X^*$ be a derivation. As above, we khnow that $X \oplus_1 \mathcal{A}$ is a Banach algebra and obviously the map

$$\varphi : a \mapsto (o, a), \qquad \mathcal{A} \longrightarrow X \oplus_1 \mathcal{A}$$

is an injective Banach algebra homomorphism. Then $H^1(\mathcal{A}, ((X \oplus_1 \mathcal{A})_\varphi)^*) = \{o\}$. We define $D_1 : \mathcal{A} \longrightarrow (X \oplus_1 \mathcal{A})^*$ by $D_1(a) = (D(a), o)$. For $a, b \in \mathcal{A}$ we have

$$\begin{aligned} D_1(ab) &= (D(ab), 0) = (D(a)b + aD(b), 0) \\ &= (D(a), 0)(0, b) + (0, a)(D(b), 0) \\ &= D_1(a)\varphi(b) + \varphi(a)D_1(b). \end{aligned}$$

Thus D_1 is a derivation from $\mathcal{A}$ into $((X \oplus_1 \mathcal{A})_\varphi)^*$. Also for every $a, b \in \mathcal{A}$, we have

$$\langle D_1(a), \varphi(b)\rangle + \langle D_1(b), \varphi(a)\rangle = \langle (D(a), 0), (0, b)\rangle + \langle (D(b), 0), (0, a)\rangle = 0.$$

Then D_1 is inner derivation. On the other word there exist $a' \in \mathcal{A}^*, x' \in X^*$ such that $D_1 = \delta_{(x', a')}$. For every $a \in \mathcal{A}$ we have

$$\begin{aligned} (D(a), o) = D_1(a) &= \delta_{(x', a')}(a) \\ &= \varphi(a)(x', a') - (x', a')\varphi(a) \\ &= (0, a)(x', a') - (x', a')(0, a) \\ &= (ax' - x'a, aa' - a'a). \end{aligned}$$

Thus $D = \delta_{x'}$. So $\mathcal{A}$ is amenable. To prove $(v) \Longrightarrow (i)$, let X be a Banach $\mathcal{A}$-bimodule and let $D : \mathcal{A} \longrightarrow X^{**}$ be a derivation. If $\varphi : \mathcal{A} \longrightarrow X \oplus_1 \mathcal{A}$ is the above injective Banach algebra homomorphism, then it is easy to show that $\varphi^{**} : \mathcal{A}^{**} \longrightarrow (X \oplus_1 \mathcal{A})^{**}$ the second transpose of φ is a Banach algebra homomorphism and that $((X \oplus_1 \mathcal{A})_\varphi)^{**} \simeq (X^{**} \oplus_1 \mathcal{A}^{**})_{\varphi^{**}}$ as $\mathcal{A}^{**}$-bimoduls. Then

$$H^1(\mathcal{A}, (X^{**} \oplus_1 \mathcal{A}^{**})_{\varphi^{**}}) = H^1(\mathcal{A}, ((X \oplus_1 \mathcal{A})_\varphi)^{**}) = \{o\} \qquad (1).$$

Now we define $D_1 : \mathcal{A} \longrightarrow X^{**} \oplus_1 \mathcal{A}^{**}$ by $D_1(a) = (D(a), o)$. For $a, b \in \mathcal{A}$ we have

$$D_1(ab) = D_1(a)\varphi^{**}(\hat{b}) + \varphi^{**}(\hat{a})D_1(b).$$

Thus D_1 is a derivation from $\mathcal{A}$ into $(X^{**} \oplus_1 \mathcal{A}^{**})_{\varphi^{**}}$. By (1), D_1 is inner. Therefore there exist $a'' \in \mathcal{A}^{**}, x'' \in X^{**}$ such that $D_1 = \delta_{(x'', a'')}$, and by a similar proof as above we can show that D is inner. Then we have $H^1(\mathcal{A}, X^{**}) = \{o\}$, and by Proposition 2.8.59 of [2], $\mathcal{A}$ is amenable. ■

Let $\mathcal{A}$ has a bounded approximate identity, and let X be an essential Banach $\mathcal{A}$-bimodule, then it is easy to show that $(X \oplus_1 \mathcal{A})_\varphi$ is an essential Banach $\mathcal{A}$-bimodule when $\varphi : \mathcal{A} \longrightarrow X \oplus_1 \mathcal{A}$ defined by $\varphi(a) = (o, a)$. By a same technique as above and by using Corollary 2.9.28 of [2], we have the following

Theorem 2.2. Let A be a Banach algebra with a bounded approximate identity. Then $\mathcal{A}$ is amenable if and only if for every Banach algebra $\mathcal{B}$ and every injective homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ in which $\mathcal{B}_\varphi$ is essential, $H^1(\mathcal{A}, {\mathcal{B}_\varphi}^*) = \{o\}$.

3. Weak amenability

In this section we fined the relationship between weak amenability and homomorphisms of Banach algebras. First we introduce two new notations of amenability of Banach algebras.

Definition 3.1. Let $\mathcal{A}$ be a Banach algebra. Then

(i) $\mathcal{A}$ is supper weakly amenable if for every Banach algebra $\mathcal{B}$ and every continuous homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$, if d_φ is a (bounded) derivation from $\mathcal{A}$ into ${\mathcal{B}_\varphi}^*$, then the following condition holds

$$\langle d_\varphi(a), \varphi(b)\rangle + \langle d_\varphi(b), \varphi(a)\rangle = 0 \qquad (a, b \in \mathcal{A}) \qquad (1).$$

(iii) $\mathcal{A}$ is semiweakly amenable if every derivation $D : \mathcal{A} \longrightarrow \mathcal{A}^*$, by the following property

$$\langle D(a), b\rangle + \langle D(b), a\rangle = 0 \qquad (a, b \in \mathcal{A}) \qquad (2),$$

is an inner derivation.

Example 1. Let $\mathbb{T}$ be the unit circle. We write $(\hat{f}(n) : n \in \mathbb{Z})$ for the sequence of Fourier coefficients of a function $f \in L^1(\mathbb{T})$. For $\alpha \in (\frac{1}{2}, 1)$, let $\mathcal{A} = lip_\alpha(\mathbb{T})$, we define $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ by

$$\langle D(f), g\rangle = \sum n\hat{g}(n)\hat{f}(-n), \qquad (f, g \in \mathcal{A}).$$

D is a non-inner derivation (see [1]) and we have

$$\langle D(f), g\rangle + \langle D(g), f\rangle = 0 \qquad (f, g \in \mathcal{A}).$$

Thus $\mathcal{A}$ is not semiweakly amenable.

Theorem 3.2. Let $\mathcal{A}$ be a supper weakly amenable Banach algebra. Then

(i) $\mathcal{A}$ is essential.

(ii) There are no no-zero continuous point derivations on $\mathcal{A}$.

Proof. (i): Let $a_0 \in \mathcal{A} - \bar{\mathcal{A}}^2$, then by Hahn-Banach theorem there exists $f \in \mathcal{A}^*$ such that $\langle f, a_0\rangle = 1$ and $f(\bar{\mathcal{A}}^2) = \{0\}$. The mapping $D : a \longmapsto f(a)f, \quad \mathcal{A} \longrightarrow \mathcal{A}^*$ is a derivation and we have $\langle D(a_0), a_0\rangle + \langle D(a_0), a_0\rangle = 2 \neq 0$. Thus $\mathcal{A}$ is not supper weakly amenable. (ii): Let $\varphi \in \Omega_\mathcal{A}$. If $\varphi = 0$, then by (i), every derivation from $\mathcal{A}$ into ${\mathbb{C}_\varphi}^*$ is zero. If $\varphi \neq 0$, and $d_\varphi : \mathcal{A} \longrightarrow \mathbb{C}_\varphi$ is a point derivation at φ, then by Definition 3.1, for every $a \in \mathcal{A}$ we have $\langle d_\varphi(a), \varphi(a)\rangle = d_\varphi(a)\varphi(a) = 0$. Therefor we have $d_\varphi|(\mathcal{A} \setminus M_\varphi) = 0$. Thus $d_\varphi = 0$. ■

Example 2. Let $\mathcal{A} = \mathbb{C}$ by the product $ab = 0, (a, b \in \mathbb{C})$. Then by Theorem 3.2 (i), $\mathcal{A}$ is not supper weakly amenable. But it is easy to check that $\mathcal{A}$ is semiweakly amenable.

Example 3. Let S be a discrete semigroup in which $S^2 \neq S$, then by Theorem 3.2 (i), $l^1(S)$ is not supper weakly amenable. Let $S = \{t, 0\}$ by products $t0 = 0t = t^2 = 0^2 = 0$, then $l^1(S)$ is not supper weakly amenable but for every derivation $D : l^1(S) \longrightarrow l^1(S)^*$ if $\langle D(\delta_t), \delta_0\rangle + \langle D(\delta_0), \delta_t\rangle = 0$, then we have $D = 0$. Thus $l^1(S)$ is semiweakly amenable.

Now we fined an equivalent condition for weak amenability of Banach algebras.

Theorem 3.3. Let $\mathcal{A}$ be a Banach algebra. Then

(i) $\mathcal{A}$ is weakly amenable if and only if $\mathcal{A}$ is supper weakly amenable and semiweakly amenable.

(ii) Let $\mathcal{A}$ be a unital Banach algebra then $\mathcal{A}$ is supper weakly amenable if and only if for every derivation $D : \mathcal{A} \longrightarrow \mathcal{A}^*$, and for every $a \in \mathcal{A}$, we have $\langle D(a), 1\rangle = 0$.

Proof. (i): Let $\mathcal{A}$ be weakly amenable. Obviously $\mathcal{A}$ is semiweakly amenable. For Banach algebra $\mathcal{B}$ and for (continuous) homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$, let $d_\varphi : \mathcal{A} \longrightarrow {\mathcal{B}_\varphi}^*$ be a derivation. We define $D = d_\varphi \otimes \varphi : \mathcal{A} \longrightarrow \mathcal{A}^*$ as follows

$$\langle D(a), b\rangle = \langle d_\varphi(a), \varphi(b)\rangle \qquad (a, b \in \mathcal{A}) \qquad (3).$$

For every $a, b, c \in \mathcal{A}$, we have

$$\begin{aligned}
\langle D(ab), c\rangle &= \langle d_\varphi(ab), \varphi(c)\rangle \\
&= \langle d_\varphi(a)\varphi(b), \varphi(c)\rangle + \langle \varphi(a)d_\varphi(b), \varphi(c)\rangle \\
&= \langle d_\varphi(a), \varphi(b)\varphi(c)\rangle + \langle d_\varphi(b), \varphi(c)\varphi(a)\rangle \\
&= \langle d_\varphi(a), \varphi(bc)\rangle + \langle d_\varphi(b), \varphi(ca)\rangle \\
&= \langle D(a), bc\rangle + \langle D(b), ca\rangle \\
&= \langle D(a)b + aD(b), c\rangle.
\end{aligned}$$

Therefore D is a derivation. Then there exists $f \in \mathcal{A}^*$ such that $D = \delta_f : \mathcal{A} \longrightarrow \mathcal{A}^*$. Thus for every $a, b \in \mathcal{A}$, we have

$$\begin{aligned}
\langle D(a), b\rangle + \langle D(b), a\rangle &= \langle \delta_f(a), b\rangle + \langle \delta_f(b), a\rangle \\
&= \langle af - fa, b\rangle + \langle bf - fb, a\rangle \\
&= 0.
\end{aligned}$$

So $\mathcal{A}$ is supper weakly amenable. The converse is trivially since $id : \mathcal{A} \longrightarrow \mathcal{A}$ is a homomorphism in which $\mathcal{A}^* = {\mathcal{A}_{id}}^*$. (ii): Let for every derivation $D : \mathcal{A} \longrightarrow \mathcal{A}^*$, and for every $a \in \mathcal{A}$, the equality $\langle D(a), 1\rangle = 0$ holds, and let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a Banach algebra homomorphism. If $d_\varphi : \mathcal{A} \longrightarrow {\mathcal{B}_\varphi}^*$ is a derivation, then $D = d_\varphi \otimes \varphi : \mathcal{A} \longrightarrow \mathcal{A}^*$ defined in the proof of (i), is a derivation and for every $a, b \in \mathcal{A}$, we have

$$\langle d_\varphi(a), \varphi(b)\rangle + \langle d_\varphi(b), \varphi(a)\rangle = \langle D(a), b\rangle + \langle D(b), a\rangle = \langle D(ab), 1\rangle = 0.$$

The converse is trivial. ∎

Corollary 3.4. (Theorem 2.8.63 of [2]) Let $\mathcal{A}$ be a weakly amenable Banach algebra, then $\mathcal{A}$ is essential and there are no non-zero, (continuous) point derivations on $\mathcal{A}$.

Corollary 3.5. Let G be a locally compact topological group, Then G is discrete if and only if $M(G)$ is supper weakly amenable.

Proof. H. G. Dales, F. Ghahramani and A. Ya. Helmeskii [3] showed that G is discrete if and only if there are no nonzero point derivations on $M(G)$. By applying Theorems 3.2 (ii) and 3.3 (i), we conclude that G is discrete if and only if $M(G)$ is supper weakly amenable. ■

By the following Theorem we can show that the supper weak amenability is different from the weak amenability and semiweak amenability.

Theorem 3.6. Let $\mathcal{A}$ be a supper weakly amenable Banach algebra, and let $\theta : \mathcal{A} \longrightarrow \mathcal{B}$ be a continuous Banach algebra homomorphism with dense range. Then $\mathcal{B}$ is supper weakly amenable.

Proof. Let $\varphi : \mathcal{B} \longrightarrow \mathcal{C}$ be a Banach algebra homomorphism and let $d_\varphi : \mathcal{B} \longrightarrow {\mathcal{C}_\varphi}^*$ be a derivation. Then for every $a, b \in \mathcal{A}$, we have

$$d_\varphi o\theta(ab) = d_\varphi o\theta(a)\varphi o\theta(b) + \varphi o\theta(a) d_\varphi o\theta(b).$$

Therefore $d_\varphi o\theta$ is a derivation from $\mathcal{A}$ into $(\mathcal{C}_{\varphi o\theta})^*$. Since $\mathcal{A}$ is supper weakly amenable, then for every $a, b \in \mathcal{A}$, we have

$$\langle d_\varphi o\theta(a), \varphi o\theta(b)\rangle + \langle d_\varphi o\theta(b), \varphi o\theta(a)\rangle = 0.$$

Since $\theta(\mathcal{A})$ is dense in $\mathcal{B}$, then for every $a', b' \in \mathcal{B}$,

$$\langle d_\varphi(a'), \varphi(b')\rangle + \langle d_\varphi(b'), \varphi(a')\rangle = 0.$$

Thus $\mathcal{B}$ is supper weakly amenable. ■

Corollary 3.7. There exists a supper weakly amenable, non-semiweakly amenable Banach algebra.

Proof. Let E be Banach space without approximation property and take $\mathcal{A}$ to be the nuclear algebra $E\hat{\otimes}E^*$ (see Definition 2.5.4 of [2]). The identification of $E \otimes E^*$ with $\mathcal{F}(E)$ extends to an epimorphism $R : E\hat{\otimes}E^* \longrightarrow \mathcal{N}(E)$ (see Theorem 2.5.3 of [2]). Set $K = kerR$, then by Corollary 2.8.43 of [2], $\mathcal{A}$ is biprojective and hence weakly amenable. If $dimK \geq 2$ then K does not have trace extension property. So by Proposition 2.8.65 (iii) of [2], $\mathcal{N}(E) = \frac{\mathcal{A}}{K}$ is not weakly amenable. On the other hand by (i) of Theorem 3.3, above, $\mathcal{A}$ is supper weakly amenable and by Theorem 3.6, $\mathcal{N}(E) = \frac{\mathcal{A}}{K}$ is supper weakly amenable. Thus by Theorem 3.3, $\mathcal{N}(E)$ is supper weakly amenable, non-semiweakly amenable Banach algebra. ■

We finish this section with a Theorem about semiweak amenability of unitization of Banach algebras, and its application to finding an example of non-supper weakly amenable Banach algebra which its unitization is supper weakly amenable.

Theorem 3.8. Let $\mathcal{A}$ be a Banach algebra. If $\mathcal{A}^\sharp$ (the unitization of $\mathcal{A}$) is semiweakly amenable, then $\mathcal{A}$ is semiweakly amenable.

Proof. Let $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ be a derivation in which (2) holds. We define $D^\sharp : \mathcal{A}^\sharp \longrightarrow {\mathcal{A}^\sharp}^*$ as follows

$$\langle D^\sharp(a, c), (b, c')\rangle = \langle D(a), b\rangle \qquad (a, b \in \mathcal{A}, c, c' \in \mathbb{C}).$$

Then for every $a, b, d \in \mathcal{A}$ and $c, c', c'' \in \mathbb{C}$, we have

$$\begin{aligned}\langle D^\sharp((a,c)(b,c')),(d,c'')\rangle &= \langle D^\sharp(ab+cb+c'a,cc'),(d,c'')\rangle = \langle D(ab+cb+c'a),d\rangle \\ &= \langle D(a)b+aD(b)+cD(b)+c'D(a),d\rangle \\ &= \langle D(a),bd+c''b+c'd\rangle + \langle D(b),da+cd+c''a\rangle \\ &= \langle D^\sharp(a,c),(bd+c''b+c'd,c'c'')\rangle + \langle D^\sharp(b,c'),(da+cd+c''a,cc'')\rangle \\ &= \langle (D^\sharp(a,c))(b,c'),(d,c'')\rangle + \langle (a,c)(D^\sharp(b,c')),(d,c'')\rangle.\end{aligned}$$

Thus $D^\sharp$ is a derivation. So we have

$$\langle D^\sharp(a,c),(b,c')\rangle = \langle D(a),b\rangle = \langle D(b),a\rangle = \langle D^\sharp(b,c'),(a,c)\rangle \qquad (a,b\in\mathcal{A}, c,c'\in\mathbb{C}).$$

$\mathcal{A}^\sharp$ is semiweakly amenable, then there is $u' \in {\mathcal{A}^\sharp}^*$ such that $D^\sharp = \delta_{u'}$. So we have $D = \delta_{(u'|_\mathcal{A})}$. Thus $\mathcal{A}$ is semiweakly amenable. ■

Let $\mathcal{A}$ be the augmentation ideal of $L^1(PS(2,\mathbb{R}))$, then we know that $\mathcal{A}^\sharp$, is weakly amenable and that $\mathcal{A}$ is not weakly amenable (see [8]). By above Theorem, $\mathcal{A}$ is semiweakly amenable. So by Theorem 3.3, $\mathcal{A}^\sharp$ is supper weakly amenable and $\mathcal{A}$ is not supper weakly amenable. Thus we have the following.

Corollary 3.9. There exists a semiweakly amenable Banach algebra $\mathcal{A}$, Such that $\mathcal{A}^\sharp$ is supper weakly amenable, and $\mathcal{A}$ is not supper weakly amenable.

4. SUPPER WEAK AMENABILITY OF THE SECOND DUAL OF BANACH ALGEBRAS

Let $\mathcal{A}^{**}$ be the second dual of $\mathcal{A}$ with the first Arens product. Then amenability of $\mathcal{A}^{**}$ implies the amenability of $\mathcal{A}$ (see for example Proposition 2.8.59 of [2]). So weak amenability of $\mathcal{A}^{**}$ implies the weak amenability of $\mathcal{A}$ if one of the following conditions holds (see [4], [5] and [6]).

(i) $\mathcal{A}$ is a left ideal in $\mathcal{A}^{**}$.

(ii) $\mathcal{A}$ is a dual Banach algebra.

(iii) $\mathcal{A}$ is Arens regular and every derivation from $\mathcal{A}$ into its dual is weakly compact.

Similarly for supper weak amenability we have

Theorem 4.1. Let $\mathcal{A}$ be a Banach algebra with one of the conditions (i), (ii) or (iii) as above. Let $\mathcal{A}^{**}$ be supper weakly amenable, then $\mathcal{A}$ is supper weakly amenable.

Proof. Let $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ be a derivation, then D has an extension $\bar{D} : \mathcal{A}^{**} \longrightarrow (\mathcal{A}^{**})^*$ in which $\bar{D}$ is a derivation (see [4], [5] and [6]). Since $\mathcal{A}^{**}$ is supper weakly amenable, then for every $a, b \in \mathcal{A}$, we have

$$\langle D(a),b\rangle + \langle D(b),a\rangle = \langle \bar{D}(\hat{a}),\hat{b}\rangle + \langle \bar{D}(\hat{b}),\hat{a}\rangle = 0.$$

Now let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a Banach algebra homomorphism and let $d_\varphi : \mathcal{A} \longrightarrow {\mathcal{B}_\varphi}^*$ be a derivation. Set $D = d_\varphi \otimes \varphi$ (defined in the proof of Theorem 3.3). We have

$$\langle d_\varphi(a), \varphi(b)\rangle + \langle d_\varphi(b), \varphi(a)\rangle = \langle D(a), b\rangle + \langle D(b), a\rangle = 0.$$

Thus $\mathcal{A}$ is supper weakly amenable. ■

Theorem 4.2. Let $\mathcal{A}$ be a Banach algebra. Let $\mathcal{A}^{**}$ be supper weakly amenable, then
(i) $\mathcal{A}$ is essential.
(ii) There are no no-zero continuous point derivations on $\mathcal{A}$.

Proof. (i): By Theorem 3.2 (i), $\mathcal{A}^{**}$ is essential, then we can show that $\mathcal{A}$ is essential (see for example Proposition 2.1 of [5]). (ii): Let $\varphi \in \Omega_{\mathcal{A}}$. If $\varphi = 0$, then by (i), every derivation from $\mathcal{A}$ into ${\mathbb{C}_\varphi}^*$ is zero. Now let $0 \neq \varphi \in \Omega_{\mathcal{A}}$, then it is easy to show that $\varphi'' \in \Omega_{\mathcal{A}^{**}}$. We suppose that $d_\varphi : \mathcal{A} \longrightarrow \mathbb{C}_\varphi$ is a point derivation at φ. Let $a'', b'' \in \mathcal{A}^{**}$ then there are nets (a_α) and (b_β) in $\mathcal{A}$ such that converge respectively to a'' and b'' in the $weak^*$- topology of $\mathcal{A}^{**}$. Then we have

$$\begin{aligned} (d_\varphi)''(a''b'') &= weak^* lim_\alpha lim_\beta d_\varphi(a_\alpha b_\beta) \\ &= weak^* lim_\alpha lim_\beta d_\varphi(a_\alpha)\varphi(b_\beta) + weak^* lim_\alpha lim_\beta \varphi(a_\alpha) d_\varphi(b_\beta) \\ &= (d_\varphi)''(a'').\varphi''(b'') + \varphi''(a'').(d_\varphi)''(b''). \end{aligned}$$

Thus $(d_\varphi)'' : \mathcal{A}^{**} \longrightarrow {\mathbb{C}_{\varphi''}}^{**}$ is a derivation. By Theorem 3.2 (ii), for every $a, b \in \mathcal{A}$, we have

$$\langle d_\varphi(a), \varphi(b)\rangle + \langle d_\varphi(b), \varphi(a)\rangle = \langle (d_\varphi)''(\hat{a}), \varphi''(\hat{b})\rangle + \langle (d_\varphi)''(\hat{b}), \varphi''(\hat{a})\rangle = 0. \qquad ■$$

References

[1] W. G. Bade, P. C. Curtis and H. G. Dales, Amenability and weak amenability for Beurling and Lipschitz algebras, Proc. London Math. Soc. 55 (1987), 359-377.

[2] H. G. Dales, Banach algebras and automatic continuity, Oxford, New York, 2000.

[3] H. G. Dales, F. Ghahramani and A. Ya. Helmeskii, The amenability of mesure algebras, Journal of the London Mathematical Society, (2) 66 (2002), 213-226.

[4] H. G. Dales, A. Rodriguez-Palacios and M. V. Velasco, The second transpose of a derivation, J. London Math. Soc. (2) 64 (2001) 707-721.

[5] F. Ghahramani and J. Laali, Amenability and topological centers of the second duals of Banach algebras, Bul. Austral. Math. soc. Vol. 65 (2002) 191-197.

[6] F. Ghahramani, R. J. Loy and G. A. Willis, Amenability and weak amenability of second conjugate Banach algebras, Proc. Amer. Math. Soc. Vol. 124, Num. **5** (1996), 1489-1497.

[7] B. E. Johnson, Cohomology in Banach algebras, Mem. Amer. Math Soc. **127** (1972).

[8] B. E. Johnson and M. C. White, A non-weakly amenable augmentation ideal, to appear .

Department of Mathematics, University of Semnan, Semnan, Iran

E-mail address: `maj_ess@Yahoo.com and meshaghi@semnan.ac.ir`